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The motions of a floating slender torus

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The motions of a floating torus oscillating in response to incident waves are analysed under the assumptions that the incident wavelength is comparable with the radius of the body section and small compared with the larger radius of the torus. This problem serves to illustrate certain features of the strip theory for ship motions, but the axisymmetric geometry and absence of body ends greatly simplify the analysis. Matched asymptotic expansions are used, with the inner solution close to the body section composed of suitable radiation and scattering problems for the two-dimensional circular cylinder. Resonant standing-wave modes in the internal basin have a singular effect upon the hydrodynamic forces acting on the body, and its response to incident waves.

1. Introduction

For ships and other floating elongated bodies, it is common to analyse the oscillatory motions in waves from a slender-body approximation, based on the disparate magnitudes of the body length and transverse dimensions. This approach has developed in part from the classical slender-body theory of aerodynamics, with the added complexity of free-surface waves. The characteristic wavelength is specially relevant, in relation to the two disparate length scales of the body. If the wavelength λ is comparable with the body length L, the wave effects are three-dimensional but free-surface effects are absent from the 'inner' flow near each section of the body. Alternatively, if the wavelength is comparable with the transverse body dimensions and small compared with L, a strip theory results with the leading-order inner solution at each body section that of a two-dimensional wave–body interaction. A recent survey of this topic is given by Ogilvie (1977).

From the practical standpoint the short-wavelength strip-theory regime is most useful, particularly for ships moving forwards, for which the Doppler shift can significantly increase the frequency ω of oscillations. Moreover, the vertical motions are resonant at natural frequencies which can be estimated by equating the hydrostatic restoring force to the inertial force, where the latter is the product of ω^2 and the effective mass. Since both forces are proportional to the length, the resonant frequencies and wavelengths are governed only by the transverse dimensions of the body. From dimensional analysis it follows that resonance will coincide with the short-wavelength regime where λ is comparable with the transverse body dimensions.

In the short-wavelength regime the body ends are of singular importance, particularly in the scattering of incident plane waves. More generally, 'end effects' are a source of inaccuracies or inconsistencies in slender-body theories which often are ignored. Justification usually rests with an assumption that the body ends are suitably pointed, but the geometric restrictions imposed by this assumption are rarely stated.

End effects are absent in the case of a slender body whose 'longitudinal' axis is a closed curve without ends. The canonical example is a torus, with circular sections of *minor* radius a, centred upon a larger circle of *major* radius c. The torus is slender in the asymptotic sense if $a \ll c$. The hydrodynamic characteristics of such a body shape are simplified not only by the lack of end effects, but also by the axisymmetric geometry. In the case of an unbounded ideal fluid, the slender torus has been analysed by Wu & Yates (1976) to establish the leading-order effects in slender-body theory associated with a curvilinear body axis.

The present paper treats the problem of a slender torus floating on the free surface, in the presence of incident plane waves. The short-wavelength regime is considered, where $\lambda/a = O(1)$ and $\lambda/c \ll 1$. This problem is analogous to that of the usual strip theory for ship motions, but the axisymmetric geometry and absence of body ends permit a relatively simple solution in terms of the corresponding two-dimensional solution for a floating circular cylinder.

Our results can be applied to any large floating structure where the buoyant volume is concentrated along the periphery of a closed curve. Possible applications might be to a circular floating breakwater or oil-containment barrier, but no existing structures of this type are known. Toroidal buoys of *small* major radius are used commonly for oceanographic purposes, but since $\lambda \gg c$ the buoy moves in a quasi-hydrostatic manner. Thus there are no direct practical applications of the present results to the author's knowledge.

In the presence of plane progressive incident waves of small amplitude, the torus will perform small oscillatory motions with the same frequency in the horizontal direction (surge), in the vertical direction (heave) and about an axis perpendicular to these two directions (pitch). In the linear theory this motion can be decomposed into three radiation problems of forced oscillations in each mode, in otherwise calm water, plus a diffraction problem of waves incident upon the stationary body. Combining the hydrodynamic pressure forces from each of these problems yields linearized equations of motion for the oscillations of the body in waves.

The three radiation problems are solved by simple applications of the method of matched asymptotic expansions. Known results for the oscillatory motion of a floating circular cylinder are used to represent the locally two-dimensional flow near the body in the inner region. Two outer regions, consisting of the interior 'basin' enclosed by the torus and the exterior domain, require separate three-dimensional analyses. Matching is applied to the amplitudes and phases of the waves propagating in the two overlap regions between these three separate domains. In view of the assumptions that $a/\lambda = O(1)$ and $a/c \ll 1$, suitable overlap regions exist where the distance from the body surface is large compared with the wavelength and minor radius a but small compared with the major radius c.

An important feature of the outer flow in the interior basin is the presence of radiated waves which propagate across the basin, and which appear as incident waves on the opposite side of the torus. The same feature is displayed in a similar two-dimensional analysis carried out by Ohkusu (1974) for a 'catamaran' configuration consisting of two long parallel cylinders, and in an exact numerical solution of the same problem by Wang & Wahab (1971).

The superposition of radiated waves travelling across the basin in opposite directions creates a circular standing-wave system. Since the transmission coefficient beneath the

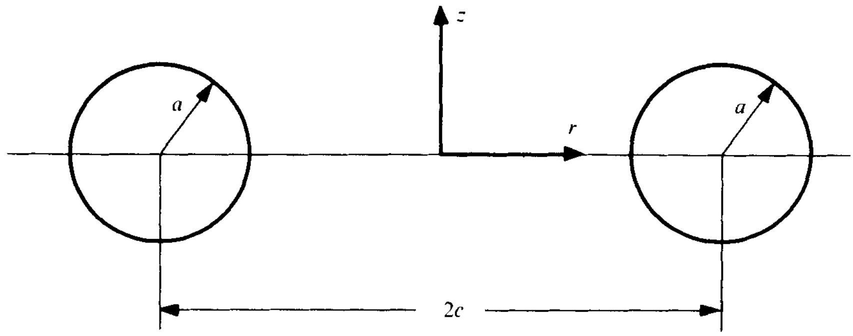

FIGURE 1. Co-ordinate system and geometry of the torus in a section θ = constant.

submerged portion of the body generally is small, these standing-wave modes are highly resonant. In this respect the problem at hand is analogous to the 'bottomless harbour' treated by Garrett (1970).

As in the two-dimensional analysis of Ohkusu, we rely on Green's theorem to relate the radiation and diffraction problems. Thus, from the Haskind relations, the exciting force due to incident waves can be evaluated in terms of the far-field wave amplitude of the radiation problem. The matching procedure is greatly facilitated by the use of additional relations, between the reflexion and transmission coefficients of the scattering problem and the far-field phase of the radiated waves. Indeed, the existence of these additional relations can be inferred by the superposition of radiation and diffraction problems, as noted in the derivation by Newman (1975). An earlier derivation of the same results by Bessho (1965) has been brought to the author's attention by Professor Ohkusu.

Circular cylindrical co-ordinates (r, θ, z) are employed, with $z = 0$ the plane of the undisturbed free surface and z positive upwards, as shown in figure 1. The direction $\theta = 0$ is chosen to coincide with the incident-wave propagation. The body is formed by rotating about the vertical (z) axis a circle of radius a centred in the plane $z = 0$ at a distance $r = c$ from the vertical axis, where $a \ll c$. Deep water is assumed for simplicity; the extension to finite depths is straightforward.

A subscript j is used to denote translation ($j = 1, 2, 3$) and rotation ($j = 4, 5, 6$) with respect to the Cartesian co-ordinates $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, $x_3 = z$. With this convention the three radiation problems to be considered are surge ($j = 1$), heave ($j = 3$) and pitch ($j = 5$). The axisymmetric case of forced heaving motion is treated in § 2, and the corresponding derivations for surge and pitch are outlined in §§ 3–4. The exciting force and moment are analysed in § 5, and these are used to form equations of motion for the body response to the incident waves.

A complementary problem for the floating torus is that of Davis (1975), who treats forced vertical motions with the ratio a/c arbitrary, but with the wavelength short compared with *both* radii. In § 2, it will be shown that the present results are consistent with those obtained by Davis (1975) in the mutual limit $\lambda \ll a \ll c$.

2. Forced heaving motions

The body is given a vertical motion with complex velocity $v_3 = i\omega\xi_3 e^{i\omega t}$. The resulting flow is axisymmetric, and may be described by a velocity potential of the form

$$\phi = \mathrm{Re}\,(v_3 \phi_3). \tag{2.1}$$

Here ϕ_3 is governed by Laplace's equation in the fluid domain, subject to the boundary condition

$$\partial\phi_3/\partial n = n_z \tag{2.2}$$

on the body surface. The unit normal vector $\mathbf{n}$ is defined as positive into the body, and n_z denotes its vertical component.

The linearized free-surface condition

$$K\phi_3 - \partial\phi_3/\partial z = 0 \tag{2.3}$$

holds on the plane $z = 0$, where $K = \omega^2/g$. At large distances exterior to the body, a radiation condition is imposed in the form

$$\phi_3 \simeq f_3(Kr)^{-\frac{1}{2}} \exp\,(Kz - iKr). \tag{2.4}$$

Thus the waves are outgoing, with complex amplitude proportional to the constant f_3.

The expression defined by (2.4) satisfies Laplace's equation to leading order for large Kr, as well as the boundary conditions of the outer problem exterior to the body. Thus (2.4) is the complete outer solution in the exterior region. Physically, this is the asymptotic form of a simple axisymmetric system of outgoing ring waves, as observed many wavelengths from the disturbance.

The corresponding outer solution in the basin interior to the body can be obtained by the method of separation of variables. Typical solutions involve the products of Bessel functions of argument kr and exponential functions of argument kz. Bessel functions of the second kind are excluded since the solution is regular at $r = 0$; for the axisymmetric heaving problem the order of the Bessel function is zero. Since the solution must satisfy the free-surface condition (2.3) and vanish for large depths, the z dependence is identical to (2.4) and the separation parameter $k = K$. Thus the complete outer solution in the interior basin is given by

$$\phi_3 = a_3 J_0(Kr)\, e^{Kz}, \tag{2.5}$$

where a_3 is a complex constant.

The inner region close to the body section extends outwards on both the interior and the exterior side over a distance of a few wavelengths. In this domain, $Kr \simeq Kc$ is asymptotically large, but $K|r-c| = O(1)$. Here the three-dimensional Laplace equation reduces to a two-dimensional form, in planes $\theta =$ constant. Formally, starting with Laplace's equation in cylindrical co-ordinates,

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{\partial^2\phi}{\partial z^2} + O\left(\frac{1}{r}\frac{\partial\phi}{\partial r}, \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right). \tag{2.6}$$

Since $r = O(c)$, whereas $(\partial/\partial r, \partial/\partial z) = O(a^{-1})$, the neglected terms in (2.6) are of order a/c and a^2/c^2, respectively, in comparison with those retained.

In the inner region it is convenient to employ Cartesian co-ordinates (x, z) in the plane θ = constant, with the origin at the centre of the body section. Thus $x = r - c$, and the governing equation is the two-dimensional Laplace equation:

$$\partial^2\phi/\partial x^2 + \partial^2\phi/\partial z^2 = 0. \tag{2.7}$$

The boundary conditions (2.2 and 2.3) are unchanged in terms of the inner frame of reference; thus the inner problem is that of a heaving circular cylinder of radius $r = a$.

Radiation conditions for the inner solution are determined by matching with the outer solutions on both sides of the body section. Exterior to the torus, the inner solution is matched with the inner limit of the outer solution (2.4):

$$\begin{aligned}\phi_3 &\simeq f_3(Kc)^{-\frac{1}{2}}\exp(Kz - iKr)\\ &= f_3(Kc)^{-\frac{1}{2}}\exp(Kz - iKx - iKc).\end{aligned} \tag{2.8}$$

Thus, as one might anticipate, the appropriate radiation condition for the inner solution on the exterior side of the torus is an outgoing two-dimensional wave motion.

The matching procedure in the interior region involves similar arguments. Using the asymptotic expansion of the Bessel function for large values of its argument, the inner limit of the outer solution (2.5) is obtained in the form

$$\phi_3 \simeq a_3(2\pi Kr)^{-\frac{1}{2}}\{\exp(Kz + iKr - \tfrac{1}{4}i\pi) + \exp(Kz - iKr + \tfrac{1}{4}i\pi)\} \tag{2.9a}$$

$$\simeq a_3(2\pi Kc)^{-\frac{1}{2}}\{\exp[Kz + iK(x+c) - \tfrac{1}{4}\pi] + \exp[Kz - iK(x+c) + \tfrac{1}{4}i\pi]\}, \tag{2.9b}$$

where (2.9*b*) is valid in the overlap region $-Kc \ll K(x-c) \ll -1$.

Equation (2.9*b*) represents a two-dimensional standing wave, the components of which can be associated with outgoing waves generated by the body section and incoming waves generated on the opposite side of the torus which propagate across the interior region. Thus the radiation condition on the inner solution for $x \to -\infty$ is given by (2.9).

The inner problem is specified by the two-dimensional Laplace equation (2.7), the body boundary condition (2.2), the free-surface condition (2.3) and by the radiation conditions (2.8 and 2.9). Since (2.9) includes an incoming wave from $x = -\infty$, the inner solution can be composed of a suitable linear combination of a scattering problem (with the same incident wave and the body fixed) and a radiation problem (with the appropriate body motion and no incident waves).

Defining the inner scattering problem in the conventional manner for incident waves of unit amplitude, the corresponding potential Φ_s satisfies the boundary condition

$$\partial\Phi_s/\partial n = 0 \tag{2.10}$$

on the body surface. At large distances from the cylinder, Φ_s is given asymptotically in the forms

$$\Phi_s \simeq \begin{cases}\exp(Kz - iKx) + R\exp(Kz + iKx) \quad \text{as} \quad x \to -\infty, & (2.11)\\ T\exp(Kz - iKx) \quad \text{as} \quad x \to +\infty. & (2.12)\end{cases}$$

Here R and T are the complex reflexion and transmission coefficients of the two-dimensional scattering problem for waves incident from $-\infty$.

In the inner radiation problem for the potential Φ_3, the cylinder oscillates with unit vertical velocity, radiating outgoing waves in a symmetric manner about $x = 0$. Thus, on the body surface,

$$\partial\Phi_3/\partial n = n_z, \tag{2.13}$$

and at large distances from the cylinder,

$$\Phi_3 \simeq F_3 \exp(Kz - iK|x|), \tag{2.14}$$

where F_3 is a complex constant.

The complete solution in the inner region is

$$\phi_3 = \Phi_3 + A_3\,\Phi_s, \tag{2.15}$$

where A_3 is a complex constant determined from (2.9) and (2.11) as

$$A_3 = a_3(2\pi Kc)^{-\frac{1}{2}} \exp(-iKc + \tfrac{1}{4}i\pi). \tag{2.16}$$

Comparison of (2.8, 2.9, 2.11, 2.12 and 2.14) gives the outgoing-wave relations

$$F_3 + RA_3 = a_3(2\pi Kc)^{-\frac{1}{2}} \exp(iKc - \tfrac{1}{4}i\pi), \tag{2.17}$$

$$F_3 + TA_3 = f_3(Kc)^{-\frac{1}{2}} e^{-iKc}. \tag{2.18}$$

Combining (2.16 and 2.17) and solving for A_3, it follows that

$$A_3 = -F_3(R + ie^{2iKc})^{-1}. \tag{2.19}$$

The last equation determines the amplitude of the incident wave in the inner solution (2.15).

The hydrodynamic pressure in the inner region is obtained by substituting the potential (2.15) in the linearized Bernoulli equation

$$p = -i\omega\rho\phi. \tag{2.20}$$

Integration over the body section gives the vertical pressure force, per unit length along the torus, and the total three-dimensional force follows by integration around the circle of radius c. Thus the vertical force due to the hydrodynamic pressure associated with vertical heaving motion of unit velocity is obtained in the form

$$\begin{aligned} -i\omega\rho \iint_S \phi_3\, n_z\, dS &= -2\pi i\omega\rho c \int_C (\Phi_3 + A_3\,\Phi_s)\, n_z\, dl \qquad (2.21)\\ &\equiv -b_{33} - i\omega m_{33}. \end{aligned}$$

Here S denotes the submerged surface of the torus, C is the profile of the body section in the plane $\theta =$ constant, b_{33} is the three-dimensional damping coefficient, which is in phase with the velocity of the torus, and m_{33} is the added mass.

The contribution from the radiation potential in (2.21) is the two-dimensional force for the heaving body section:

$$-i\omega\rho \int_C \Phi_3\, n_z\, dl \equiv -B_{33} - i\omega M_{33}. \tag{2.22}$$

The remaining contribution from the scattering potential Φ_s is proportional to the wave-exciting force acting on the fixed two-dimensional cylinder in the presence of an incident wave system. This can be related to the radiation solution by the Haskind relations (cf. Newman 1976, equation 45). Accounting for the time dependence $e^{i\omega t}$, it follows that

$$-i\int_C \Phi_s n_z\, dl = \lim_{x\to -\infty}\{\Phi_3 \exp(-Kz-iKx)\} = F_3. \tag{2.23}$$

Combining (2.21–2.23) and using (2.19), the total vertical force on the torus is

$$b_{33}+i\omega m_{33} = 2\pi c(B_{33}+i\omega M_{33})+2\pi\omega\rho c F_3^2(R+ie^{2iKc})^{-1}. \tag{2.24}$$

From energy conservation, the damping parameter B_{33} is given by

$$B_{33} = \omega\rho|F_3|^2 \tag{2.25}$$

(*ibid*., equation 31*a*). Moreover, the reflexion coefficient R can be related to the arguments of F_3 and the corresponding sway-induced radiation wave amplitude F_1 by the relation (*ibid*, equation 52; see also Newman 1975)

$$R = -\tfrac{1}{2}(F_3/F_3^* + F_1/F_1^*) \equiv -\cos\alpha e^{i\beta}, \tag{2.26}$$

where

$$\left.\begin{matrix}\alpha\\ \beta\end{matrix}\right\} = \arg F_1 \mp \arg F_3. \tag{2.27}$$

Combining (2.24)–(2.26), we obtain an expression for the three-dimensional vertical force in the form

$$b_{33}+i\omega m_{33} = 2\pi c(B_{33}+i\omega M_{33})-2\pi c B_{33}\, e^{i(\beta-\alpha)}(\cos\alpha e^{i\beta}-ie^{2iKc})^{-1}. \tag{2.28}$$

With the definition

$$\gamma = 2Kc-\beta-\tfrac{1}{2}\pi, \tag{2.29}$$

it follows that

$$b_{33}+i\omega m_{33} = 2\pi c(B_{33}+i\omega M_{33})-2\pi c B_{33}\, e^{-i\alpha}(\cos\alpha+e^{i\gamma})^{-1}. \tag{2.30}$$

Thus the three-dimensional damping and added-mass parameters of the torus have been related to the corresponding parameters for the two-dimensional cylinder and the two angles α and β which are also properties of the two-dimensional radiation problem. The multiplicative factor $2\pi c$ is the arc length over which the two-dimensional force of the inner solution is acting; except for this factor, the only effect of the torus radius c is upon γ, which accounts for the phase of the waves propagating across the interior region.

In the limiting case $Ka \gg 1$, $T\to 0$ and $\cos\alpha \equiv |R| \to 1$. Excluding the resonant frequencies $\gamma = \pi(2n+1)$, the damping and added-mass components of (2.30) are given by the asymptotic approximations

$$b_{33} \cong \pi c B_{33} \cong (16/\pi)(Ka)^{-4} m\omega, \tag{2.31}$$

$$m_{33} \cong 2\pi c M_{33} \cong [1-4/(3\pi Ka)]\, m, \tag{2.32}$$

where $m = \pi^2\rho a^2 c$ is the mass of fluid displaced by the torus and the high-frequency approximations for B_{33} and M_{33} are derived by Ursell (1953).† The added mass (2.32) can be deduced from a simple strip theory, as the product of the two-dimensional added mass and the circumference of the torus. The damping coefficient (2.31) is reduced by one-half of the corresponding strip-theory result, since the waves generated

† The higher-order term retained in (2.32) is justified since $T = O((Ka)^{-4})$, as shown by Ursell (1961).

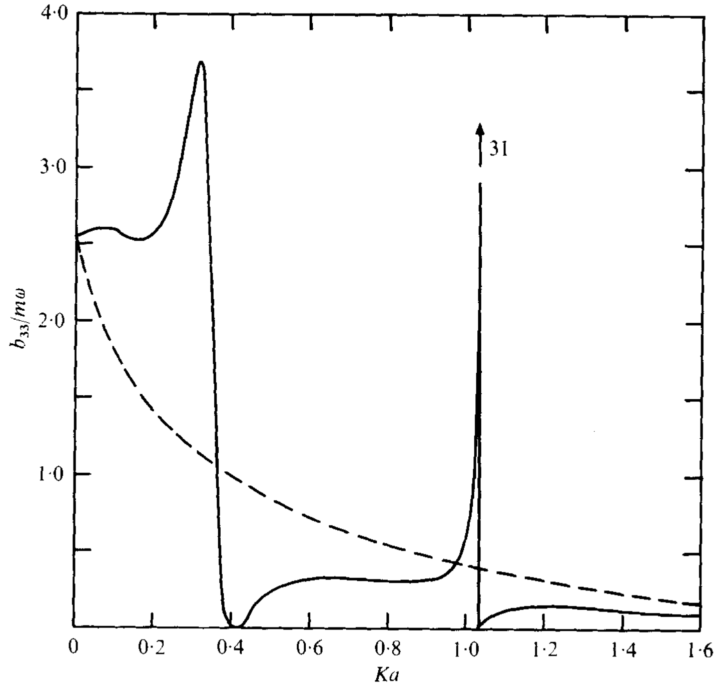

FIGURE 2. Heave damping coefficient for a torus with $a/c = 0{\cdot}2$. The dashed line denotes the corresponding two-dimensional coefficient for a circular cylinder, and the arrow denotes a peak value equal to 31.

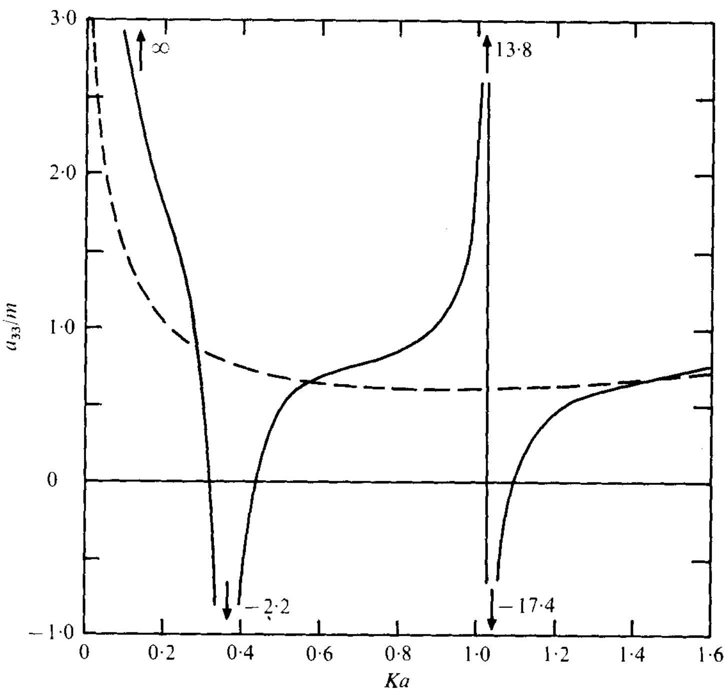

FIGURE 3. Heave added-mass coefficient for a torus with $a/c = 0{\cdot}2$. The dashed line denotes the corresponding two-dimensional coefficient for the circular cylinder, and the arrows denote peak values as shown.

on the interior side of the torus are prevented from radiating to infinity. In the limit as $Ka \to \infty$ equation (2.31) and the leading term of (2.32) are consistent with the asymptotic theory of Davis (1975) for a non-slender torus if, in Davis' results, $a/c \ll 1$.

More generally, the damping and added mass can be computed from (2.30) using the corresponding coefficients and phase angles of the two-dimensional problem for an oscillating circular cylinder. The results shown in figures 2 and 3 have been computed in this manner, using two-dimensional data provided by Lee (private communication, 1976; also 1977). Figure 2 shows the damping coefficient for a torus of slenderness ratio $a/c = 0{\cdot}2$, as well as the corresponding two-dimensional damping coefficient. Similar results for the added-mass coefficients are shown in figure 3. Both parameters display the effects of highly tuned resonant modes in the interior basin, which become increasingly sharp as the frequency increases and the transmission coefficient is decreased. The damping coefficient rises sharply, in the vicinity of each resonance, and then decreases to zero. The added mass changes abruptly from a large positive value to a large negative value in the same frequency range. The singular nature of the results shown in figures 2 and 3 is increased for smaller values of a/c, where the resonant modes are more densely spaced in terms of the parameter Ka. Increasing Ka, with a/c fixed, decreases the transmission coefficient T and increases the resonant tuning.

3. Surge motions

For forced horizontal motion, with complex velocity v_1, the velocity potential $\phi = v_1 \phi_1$ is subject to the boundary condition

$$\partial\phi_1/\partial n = n_r \cos\theta \tag{3.1}$$

on the body. Thus the potential ϕ_1 will have angular dependence proportional to $\cos\theta$. This is the principal difference relative to the axisymmetric heave problem.

In the outer region exterior to the torus (2.4) is replaced by

$$\phi_1 \simeq f_1 \cos\theta (Kr)^{-\frac{1}{2}} \exp(Kz - iKr), \tag{3.2}$$

and in the outer region interior to the torus (2.5) is replaced by

$$\phi_1 = a_1 \cos\theta\, J_1(Kr)\, e^{Kz}. \tag{3.3}$$

The inner solution is governed by the two-dimensional Laplace equation (2.7) and the body boundary condition (3.1). Exterior to the torus the inner solution is matched to the inner approximation of (3.2):

$$\phi_1 \simeq f_1 \cos\theta\, (Kc)^{-\frac{1}{2}} \exp(Kz - iKx - iKc). \tag{3.4}$$

Interior to the torus, the inner solution is matched to the inner limit obtained from (3.3) and analogous to (2.9):

$$\phi_1 \simeq -ia_1 \cos\theta (2\pi Kc)^{-\frac{1}{2}} \{\exp[Kz + iK(x+c) - \tfrac{1}{4}i\pi] - \exp[Kz - iK(x+c) + \tfrac{1}{4}i\pi]\}. \tag{3.5}$$

The appropriate inner solution is written in the form

$$\phi_1 = (\Phi_1 + A_1 \Phi_s) \cos\theta, \tag{3.6}$$

where Φ_1 is the two-dimensional radiation solution for horizontal oscillations with unit velocity. From symmetry considerations, the far-field waves of this two-dimensional potential are of the form

$$\Phi_1 \simeq \pm F_1 \exp(Kz - iK|x|) \quad \text{as} \quad x \to \pm\infty. \tag{3.7}$$

The incident wave A_1 is determined by matching the incoming wave components in (3.5) and (3.6). Thus it follows that

$$A_1 = ia_1(2\pi Kc)^{-\frac{1}{2}} \exp(-iKc + \tfrac{1}{4}i\pi). \tag{3.8}$$

Matching the outgoing wave components gives the relations

$$-F_1 + RA_1 = -ia_1(2\pi Kc)^{-\frac{1}{2}} \exp(iKc - \tfrac{1}{4}i\pi), \tag{3.9}$$

$$F_1 + TA_1 = F_1(Kc)^{-\frac{1}{2}} e^{-iKc}. \tag{3.10}$$

Combining (3.8 and 3.9), the incident wave amplitude is given by

$$A_1 = F_1(R - ie^{2iKc})^{-1}. \tag{3.11}$$

The damping and added mass follow as in the heave problem; here the factor $\cos^2\theta$ must be recalled before integrating around the torus. Thus, in place of (2.30) it follows that

$$b_{11} + i\omega m_{11} = \pi c(B_{11} + i\omega M_{11}) - \pi c B_{11} e^{i\alpha}(\cos\alpha - e^{i\gamma})^{-1}, \tag{3.12}$$

where γ is defined by (2.29).

Calculations analogous to those shown in figures 2–3 can be performed without difficulty. The results are similar, but with the zeros of the damping coefficient *preceding* the adjacent peaks, and with the resonant values of Kc shifted by π. The two-dimensional surge added mass is well behaved for $Ka \to 0$, with the (rigid free-surface) limit $M_{11} \to \frac{1}{2}\pi\rho a^2$. Thus, to leading-order,

$$m_{11} \simeq \tfrac{1}{2}\pi^2\rho a^2 c = \tfrac{1}{2}m \quad \text{as} \quad Ka \to 0, \tag{3.13}$$

in agreement with the result given by Wu & Yates (1976) for a slender torus in an unbounded fluid. In the high-frequency limit, results similar to (2.31) and (2.32) can be derived using the approximations

$$B_{11} \simeq 4\rho\omega/K^2, \tag{3.14}$$

$$M_{11} \simeq \frac{2}{\pi}\rho a^2 \left[1 - \frac{4}{Ka}\left(\frac{\pi}{9} + \frac{1}{3\pi}\right)\right] \tag{3.15}$$

for the two-dimensional damping and added mass.†

† The high-frequency approximations (3.14 and 3.15) are not so well established as the corresponding results for heave used in (2.31 and 2.32). The damping coefficient (3.14) can be obtained directly from energy conservation using the result given by Davis (1976, §2) for the radiated wave amplitude. The added mass (3.15) can be derived from Green's theorem using the procedure which is outlined for heave by Rhodes-Robinson (1971, p. 316) and Davis (1975, equation 1.10); the only modification required here is to replace the limiting infinite-frequency potential for heave by the corresponding result for sway.

4. Pitch motions

Here a rotational velocity v_5 about the axis normal to the plane $\theta = 0$ and passing through the origin $r = z = 0$ is given. The resulting velocity potential $\phi = v_5\phi_5$ is subject to the boundary condition

$$\partial\phi_5/\partial n = (zn_r - rn_z)\cos\theta \tag{4.1}$$

on the body surface. This implies, as in §3, a solution proportional to $\cos\theta$. Only the direction cosine n_z appears in the leading-order boundary condition; hence the inner solution involves the heave potential Φ_3. Physically, pitching motion of the slender torus appears locally as a heave motion with vertical velocity equal to the product of the pitch angular velocity and the radius from the pitch axis.

Outer solutions similar to (3.2) and (3.3) are applicable, with (3.4), (3.5) and (3.8) unchanged except for the subscript 5 in place of 1. The inner solution is identical to that derived in §2, except for a change in the multiplicative factors. From matching the outgoing waves, it follows that

$$F_5 + RA_5 = -ia_5(2\pi Kc)^{-\frac{1}{2}}\exp(iKc - \tfrac{1}{4}i\pi), \tag{4.2}$$

$$F_5 + TA_5 = f_5(Kc)^{-\frac{1}{2}}e^{-iKc}, \tag{4.3}$$

$$A_5 = -F_5(R - ie^{2iKc})^{-1}. \tag{4.4}$$

On computing the pressure moment as in (2.20)–(2.23) and using (2.25)–(2.27), the three-dimensional damping and added moment of inertia follow in the form

$$b_{55} + i\omega m_{55} = \pi c^3(B_{33} + i\omega m_{33}) - \pi c^3 B_{33} e^{-i\alpha}(\cos\alpha - e^{i\gamma})^{-1}. \tag{4.5}$$

Once again, the results of computations are similar to those shown in figures 2 and 3, but with a phase shift of π in the value of Kc at which resonance occurs.

5. The scattering problem

Here the body is stationary in the presence of incident waves, which propagate in the direction $\theta = 0$. If the incident wave's amplitude is denoted by A, its velocity potential is given by the real part of

$$(gA/\omega)\exp(Kz - iKr\cos\theta + i\omega t) \equiv A\phi_0 e^{i\omega t}. \tag{5.1}$$

The total potential can be written in the form

$$\phi = Ae^{i\omega t}(\phi_0 + \phi_d), \tag{5.2}$$

where ϕ_d denotes the diffraction potential due to the body. The boundary condition on the body is that the normal velocity should vanish, and thus

$$\partial\phi_d/\partial n = -\partial\phi_0/\partial n = -\omega(n_z - in_x\cos\theta)\exp[Kz - iK(x-c)\cos\theta]. \tag{5.3}$$

In all other respects the diffraction potential is governed by the same boundary-value problem as the radiation potentials ϕ_j.

Despite the similar boundary-value problems for the diffraction and radiation potentials, the solution of the diffraction problem generally is more difficult. This is attributed partially to the boundary condition (5.3), in particular the exponential factor therein, which is absent from (2.2) and (3.1). In the present case, however, a more significant complication is the lack of axisymmetry, or of the simple angular dependence proportional to $\cos\theta$. This complication can be overcome by a Fourier–Bessel expansion of the incident-wave potential (5.1), as carried out for example by Garrett (1970). That approach is expedient for the ultimate computation of the exciting force and moment, which depend exclusively on the terms with $n = 0, 1$ of the Fourier expansion in $\cos n\theta$.

Alternatively, the need to consider the scattering problem can be circumvented completely by using the three-dimensional Haskind relations to compute the exciting force and moment, and this simpler approach will be adopted below. First, however, it is instructive to consider briefly the direct solution of the diffraction problem posed above, following the approach used for the radiation problem in §§2–4. This diversion emphasizes the distinction of the torus, relative to a slender body with straight axis and finite length.

In a plane $\theta =$ constant, (5.3) is the boundary condition for diffraction of a plane wave by a two-dimensional circular cylinder. The angle between the incident-wave crests and the cylinder axis is θ, and the phase of the incident wave is $Kc\cos\theta$. In this respect, the inner problem is that of scattering of an oblique incident wave by a two-dimensional cylinder. The angle of incidence varies slowly along the torus, but since $Kc \gg 1$ the phase is a rapidly varying function.

The inner solution for the diffraction potential can be approximated as the product of a slowly varying function of θ and the oscillatory phase factor $\exp(iKc\cos\theta)$. This inner solution is governed by the two-dimensional wave equation

$$(\partial^2/\partial x^2 + \partial^2/\partial z^2 - K^2 \sin^2\theta)\,\Phi = 0 \tag{5.4}$$

in planes normal to the body section. The resulting local force vector acting on the body can be expressed in the form $\mathbf{f}(\theta)\exp(iKc\cos\theta)$, where $\mathbf{f}(\theta)$ is slowly varying. Integrating around the torus, the total exciting force is

$$\begin{aligned}\mathbf{X} &= c\int_0^{2\pi} \mathbf{f}(\theta)\, e^{iKc\cos\theta}\, d\theta \\ &\simeq c(2\pi/Kc)^{-\frac{1}{2}}[\mathbf{f}(0)\exp(iKc - \tfrac{1}{4}i\pi) + \mathbf{f}(\pi)\exp(-iKc + \tfrac{1}{4}\pi)].\end{aligned} \tag{5.5}$$

Here the stationary-phase approximation is used, on the assumption that $Kc \gg 1$. The principal contributions to the exciting force are from the points on the body where the wave is locally incident from abeam, and where the wave equation (5.4) reduces to the Laplace equation. Thus the complexity of the wave equation does not affect the exciting force to leading order in the large parameter Kc.

In the analogous scattering problem for a long slender *straight* body, of length L, the angle θ is fixed. The inner solution is oscillatory along the body length and governed by the wave equation (5.4) with *constant* wavenumber $K\cos\theta$. Excluding the case of beam waves ($\cos\theta = 0$), the local exciting force is oscillatory along the length, and the integrated force is small owing to cancellation; from the Riemann–Lebesque lemma the total exciting force can be estimated as the body mass times a factor, of order $(KL)^{-1}$, which depends solely on the body shape near the ends. If the body is pointed, the characteristic length in the inner solution tends to zero at the ends, and the (suitably non-dimensionalized) wave equation reduces to the Laplace equation at these points.

Once again the leading-order exciting force is dominated by an inner solution governed by the Laplace equation, but for entirely different reasons.

A complete inner solution of the scattering problem for the torus requires that a radiation condition on the inside of the body be derived from the inner limit of the outer solution in the interior basin. Instead of pursuing this matter here, we adopt the alternative approach to the exciting force and moment based on the three-dimensional Haskind relations.

In general, the complex amplitude X_j ($j = 1, \ldots, 6$) of the exciting force or moment is related to the wave amplitude radiated in the direction opposite to the incident wave by forced oscillations in the jth mode without incident waves. Thus, for the incident wave (5.1),

$$X_j = (\rho g A/K)\,(2\pi)^{\frac{1}{2}}\, e^{-\frac{1}{4}i\pi} f_j(\pi) \tag{5.6}$$

(cf. Newman 1976, equation 45). The analogous expression for the two-dimensional case is (2.23). The radiated wave amplitudes f_j are defined for $\theta = 0$ by the matching conditions (2.18), (3.10) and (4.3). Since the parameters on the left sides of these equations are $O(1)$ in terms of the inner solution, it follows from (5.6) that the ratio of the wave-exciting force to the body mass is of order $(Kc)^{-\frac{1}{2}}$, in agreement with the qualitative result (5.5). The corresponding result for the exciting moment, non-dimensionalized in terms of the polar moment of inertia of the body mass, is of order $(Kc)^{-\frac{3}{2}}$.

The radiated wave energy in each forced-motion problem is proportional to $|f_j|^2$, and can be related to the corresponding damping coefficient by energy conservation. Thus the magnitude of the exciting force (5.6) is proportional to the square root of the damping coefficient b_{jj}, It follows that the exciting-force components vanish at the same frequencies as the damping coefficients.

6. Body motions in waves

The body motions in waves can be determined from equations of motion relating the total pressure force to the product of the body mass m and its acceleration. There are no coupling effects between the three (linearized) modes of surge, heave and pitch, assuming the origin is located at the centre of the torus. Thus, for heave, the complex amplitude $\xi_3\, e^{i\omega t} = v_3/i\omega$ follows from the linear equation of motion

$$[-\omega^2(m_{33}+m)+i\omega b_{33}+4\pi\rho g a c]\,\xi_3 = X_3. \tag{6.1}$$

The last term in the square brackets in the hydrostatic restoring force, which is proportional to the water-plane area $4\pi ac$. Similar equations apply for surge, where the hydrostatic term is deleted, and for pitch, where the body mass m is replaced by the moment of inertia and the water-plane area is replaced by the second moment $2\pi ac^3$.

The magnitude of the body response in each mode can be calculated without difficulty from the results derived in §§2–4, with the results shown in figure 4. Each mode of body motion vanishes at the zeros of the respective damping coefficients, owing to vanishing of the corresponding exciting force, as anticipated in the closing paragraph of §5. At intermediate values of the frequency parameter resonant peaks occur when the sum of the real terms in the square brackets in (6.1) vanishes; since the added-mass coefficients take on negative values as shown in figure 3, two closely spaced resonant peaks of the body motion occur in each frequency regime where the standing waves of

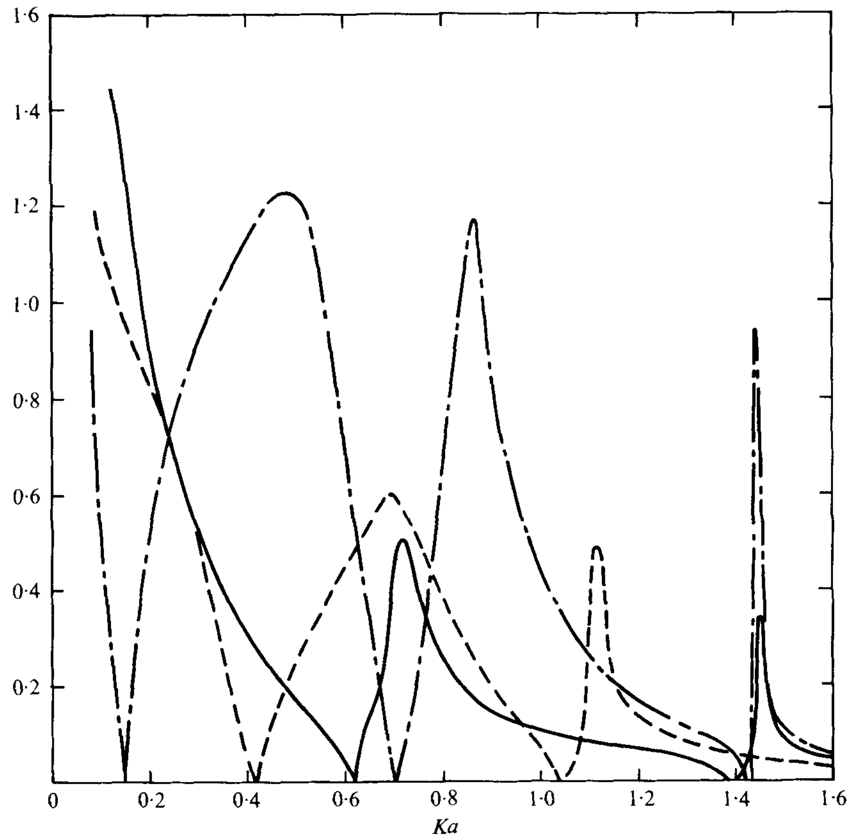

FIGURE 4. Amplitudes of body oscillations in surge (ξ_1), heave (ξ_3) and pitch (ξ_5) for torus with $a/c = 0{\cdot}2$ in incident waves of amplitude A. ——, $|\xi_1|/A$; ---, $|\xi_3|/A$; —·—, $|\xi_5|c/A$.

the internal basin are resonant. These resonant body motions do not depend on the existence of a hydrostatic restoring force or 'spring constant' as in conventional second-order oscillators or floating bodies with positive-definite added-mass coefficients, nor are the resonant frequencies limited in number.

At very low frequencies the results shown in figure 4 are invalid, because Kc is not large. Thus the present results for the oscillatory motions tend to infinity at small values of Ka, whereas in an exact three-dimensional treatment, or in a three-dimensional slender-body approach where $Kc = O(1)$, the low-frequency response is dominated by hydrostatic effects.

7. Discussion

As an example of a slender floating body in waves, the torus is unique in several respects. The axisymmetric geometry is a major simplification, which enables a short-wave asymptotic theory to be derived from simple matching arguments. The results depend principally upon the two-dimensional damping and added-mass coefficients of the circular section, and upon the phase angles of the radiated waves for the same two-dimensional body. Extension to other sectional shapes is straightforward, as is the generalization to finite fluid depth.

The absence of body ends and the curved form of the axis are significant, particularly in the scattering problem. The presence of an interior basin, with associated standing-wave modes, has a dominant effect on the hydrodynamic characteristics of the torus. In the neighbourhood of the resonant standing-wave frequencies the damping and added mass oscillate rapidly, and the added mass becomes negative. The body motions in waves are affected in a corresponding manner, as shown in figure 4.

These features may be modulated significantly by the effects of viscosity and nonlinearity, both of which are neglected. Our results are further restricted to the short-wavelength domain $Kc \gg 1$, and are not valid as $Ka \to 0$. A complementary slender-body theory can be developed for the case where $Kc = O(1)$, following the 'slender ship' theory which has been studied by various authors; the work of Ursell (1962) is relevant for an axisymmetric body. From this complementary approach one can derive results for a slender torus which are uniformly valid as $Ka \to 0$. These can be combined with the present analysis to obtain a composite theory, along the lines suggested by Maruo (1970) for a conventional ship hull.

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REFERENCES

BESSHO, M. 1965 On the theory of rolling motion of ships among waves. *Rep. Sci. Res. Defence Acad. Japan* **3**, 1.

DAVIS, A. M. J. 1975 Short surface waves due to a heaving torus. *Mathematika* **22**, 122–134.

DAVIS, A. M. J. 1976 On the short surface waves due to an oscillating, partially immersed body. *J. Fluid Mech.* **75**, 791–807.

GARRETT, C. J. R. 1970 Bottomless harbours. *J. Fluid Mech.* **43**, 433–449.

LEE, C. M. 1977 Motion characteristics of floating bodies. *J. Ship Res.* **20**, 181–190.

MARUO, H. 1970 An improvement of the slender body theory for oscillating ships with zero forward speed. *Bull. Fac. Engng, Yohakama Nat. Univ.* **19**, 45–56.

NEWMAN, J. N. 1975 Interaction of waves with two-dimensional obstacles: a relation between the radiation and scattering problems. *J. Fluid Mech.* **71**, 273–282.

NEWMAN, J. N. 1976 The interaction of stationary vessels with regular waves. *Proc. 11th Symp. Naval Hydrodyn., Univ. Coll., London.*

OGILVIE, T. F. 1977 Singular perturbation problems in ship hydrodynamics, *Adv. in Appl. Mech.* **17**, 92–187.

OHKUSU, M. 1974 Hydrodynamic forces on multiple cylinders in waves. *Proc. Int. Symp. Dyn. Marine Vehicles and Structures in Waves*. pp. 107–112. London: Inst. Mech. Engrs.

RHODES-ROBINSON, P. F. 1971 On the short cylindrical waves due to a body heaving on water. I. *Proc. Camb. Phil. Soc.* **70**, 311–321.

URSELL, F. 1953 Short surface waves due to an oscillating immersed body. *Proc. Roy. Soc.* A **220**, 90–103.

URSELL, F. 1961 The transmission of surface waves under surface obstacles. *Proc. Camb. Phil. Soc.* **57**, 638–668.

URSELL, F. 1962 Slender oscillating ships at zero forward speed. *J. Fluid Mech.* **14**, 496–516.

WANG, S. & WAHAB, R. 1971 Heaving oscillations of twin cylinders in a free surface. *J. Ship Res.* **15**, 33–48.

WU, T. Y. & YATES, G. T. 1976 Finite-amplitude unsteady slender-body flow theory. *Proc. 11th Symp. Naval Hydrodyn. Univ. Coll., London.*